An asymptotic expression for forced convection in non-Newtonian power-law fluids

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A general asymptotic expression based on the assumption of high Prandtl number, has been derived through a simple integral analysis, for a speedy and accurate estimation of the forced convection heat transfer from external surfaces to non-Newtonian power-law fluids. The local skin friction and heat transfer rate can readily be evaluated once the Falkner-Skan flow parameter, the power-law index and the exponent associated with the wall temperature distribution are specified. Comparison of the calculated results and previous numerical results on a stagnation flow over a horizontal circular cylinder reveals the validity of the present asymptotic expression.

Keywords: *power-law fluids, forced convection, boundary layer, integral analysis*

Due to the frequent use of non-Newtonian fluids in modern industry, considerable attention has been directed towards understanding the skin friction and heat transfer characteristics of the non-Newtonian fluids[1]. The present study is concerned with the boundary layer flows of the incompressible non-Newtonian fluids which can be characterized by a so-called 'power-law description'. Complexities of the problem lie in the high degree of non-linearity and coupling of the governing equations, as a direct consequence of the employment of the power-law model. Thus, similarity solutions[2,3] are not possible even in the case of a flow over an isothermal flat plate. The iterative calculation procedure proposed recently by the authors[4] has been designed for such non-similar flow problems.

It will, however, be shown that a simple integral analysis based on the von-Kármán integral relation is quite effective for investigating these complex non-Newtonian flow behaviours. Noting the fact that Prandtl number is quite large for most practical problems involving non-Newtonian fluids, a general asymptotic expression has been derived under the assumption of high Prandtl number, for a quick and accurate estimation of the forced convection heat transfer from external surfaces to the power-law fluids. The resulting expression is valid for arbitrary values of the Falkner-Skan free stream velocity exponent, the power-law index, and the exponent describing the wall temperature distribution. Calculated results on a stagnation flow over a horizontal circular cylinder are in excellent agreement with previous numerical calculation results[5].

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Analysis

The integral momentum equation may be written in the usual boundary layer coordinates (x, y) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta (u_e u - u^2)\,\mathrm{d}y + \frac{\mathrm{d}u_e}{\mathrm{d}x}\int_0^\delta (u_e - u)\,\mathrm{d}y = \frac{K}{\rho}\left(\frac{\partial u}{\partial y}\right)^n\Bigg|_{y=0} \tag{1}$$

where δ is the velocity boundary layer thickness; ρ, the density; u, the stream-wise velocity component; and the subscript e refers to at the boundary layer edge. The wall shear stress is expressed according to the power-law model introducing the two empirical constants, namely, K and the power-law exponent n. Since the inertia terms vanish at the wall, the following relation should hold:

$$-\frac{K}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n\Bigg|_{y=0} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}x} = m\frac{u_e^2}{x} \tag{2a}$$

$$u_e \propto x^m \tag{2b}$$

As indicated by Eq (2b), the Falkner-Skan free stream velocity field is assumed to prevail. A free stream velocity distribution of this type may be realized for a flow passing over an infinite wedge or cone, but is found also in the vicinity of a sharp leading edge of any closed body, where the rate of heat transfer is most significant. For the velocity profile within the boundary layer, the Pohlhausen's polynomial may be chosen:

$$u/u_e = (2+\Lambda)\eta - 3\Lambda\eta^2 - (2-3\Lambda)\eta^3 + (1-\Lambda)\eta^4 \tag{3a}$$

where

$$\eta = y/\delta \tag{3b}$$

Upon substituting Eq (3a) into Eqs (1) and (2a), and carrying out differentiations and integrations, the two distinct expressions for δ may be reduced as follows:

$$(\delta/x)^{1+n}Rex=\frac{(1+n)C^n}{G}\frac{1}{1+(3n+(1+n)H)m}$$

$$=6nC^{n-1}\Lambda/m \tag{4a}$$

where

$$C=\frac{\delta}{u_e}\frac{\partial u}{\partial y}\bigg|_{y=0}=2+\Lambda \tag{4b}$$

$$G=\int_0^\delta (u_e u-u^2)\,dy/u_e^2\delta$$

$$=(148-8\Lambda-5\Lambda^2)/1260 \tag{4c}$$

$$H=\int_0^\delta (u_e-u)\,dy/Gu_e\delta=\frac{6-\Lambda}{20G} \tag{4d}$$

and

$$Rex=\rho x^n u_e^{2-n}/K \tag{4e}$$

is the Reynolds number. The foregoing two expressions on the right hand side of Eq (4a) provide the following algebraic equation among Λ, n and m:

$$m=\frac{\Lambda G}{\frac{1+n}{6n}C-\Lambda\left(3nG+\frac{(1+n)(6-\Lambda)}{20}\right)} \tag{5}$$

Eq (5) can readily be solved for Λ, as the Falkner-Skan flow parameter m and the power-law exponent n are specified. Once Λ is determined in this way, the local skin friction coefficient $C_{fx}=2\tau_w/\rho u_e^2$ may be evaluated from

$$C_{fx}Rex^{1/(1+n)}=2\left(\frac{mC}{6n\Lambda}\right)^{n/(1+n)} \tag{6}$$

Since Prandtl numbers for actual non-Newtonian fluids are usually very large, the integral energy equation may well be approximated as

$$\frac{d}{dx}\int_0^{\delta_t}\frac{\partial u}{\partial y}\bigg|_{y=0} y(T-T_e)\,dy=-\frac{k}{\rho C_p}\frac{\partial T}{\partial y}\bigg|_{y=0} \tag{7}$$

where δ_t is the thermal boundary layer thickness which is assumed much less than δ; T, the temperature; k, the thermal conductivity; and C_p, the specific heat. Since the second derivative of the temperature profile must vanish at the wall, the temperature profile across the boundary layer is assumed to follow

$$(T-T_e)/(T_w-T_e)=1-2\eta_t+2\eta_t^3-\eta_t^4 \tag{8a}$$

where

$$\eta_t=y/\delta_t \tag{8b}$$

and

$$T_w-T_e\propto x^{m_t} \tag{8c}$$

The wall temperature is assumed to vary according to the power function of x. These proposed velocity and temperature profiles are substituted into Eq (7), and then, the following equation may be reduced after some integration:

$$(\delta_t/x)^2=\frac{60k\zeta}{\rho C_p u_e xC}\frac{1}{1+m+2m_t-\frac{d\ln\zeta}{d\ln x}} \tag{9a}$$

where

$$\zeta=\delta/\delta_t \tag{9b}$$

is the boundary layer thickness ratio. Combining Eqs (9a) and (4a), one obtains

$$\zeta\simeq\left[\left(\frac{6nC^{n-1}\Lambda}{m}\right)^{2/(1+n)}\frac{2+\Lambda}{60}\left(1+m+2m_t-\frac{(1-n)(1-3m)}{3(1+n)}\right)\right]^{1/3}Prx^{1/3} \tag{10a}$$

where

$$Prx=\frac{\rho C_p}{k}\left(\frac{K}{\rho}\right)^{2/(1+n)}\left(\frac{x}{u_e^3}\right)^{(1-n)/(1+n)} \tag{10b}$$

is the Prandtl number. Whether Prx (apparent viscosity) increases downstream or not, solely depends on the sign of $(1-3m)(1-n)/(1+n)$. Obviously, for a right angle wedge ($m=1/3$), Prx remains constant, as in Newtonian fluids ($n=1$). Since ζ is readily calculable from Eq (10a) by substituting Λ obtained from Eq (5), the local Nusselt number $Nux=2\zeta x/\delta$ of the primary concern, may be determined from

$$Nux/Rex^{1/(1+n)}\simeq\left[\left(\frac{m}{6nC^{n-1}\Lambda}\right)^{1/(1+n)}\frac{2(2+\Lambda)}{15}\left(1+m+2m_t-\frac{(1-n)(1-3m)}{3(1+n)}\right)\right]^{1/3}Prx^{1/3} \tag{11}$$

Notation

C, G, H	Boundary layer shape factors
C_{fx}	Local skin friction coefficient
C_p	Specific heat
k	Thermal conductivity
K	Multiplicative constant in the power-law model
m	Falkner-Skan flow parameter
m_t	Exponent for the wall temperature variation
n	Power-law exponent
Nux	Local Nusselt number
Prx	Prandtl number
Rex	Reynolds number
T	Temperature
u	Velocity component in the x direction
x, y	Boundary layer coordinates
δ, δ_t	Viscous and thermal boundary layer thicknesses
ζ	Boundary layer thickness ratio
η, η_t	Dimensionless variable in the y direction
Λ	Shape factor associated with the curvature at the wall

Subscripts

e	Boundary layer edge
t	Thermal boundary layer
w	Wall

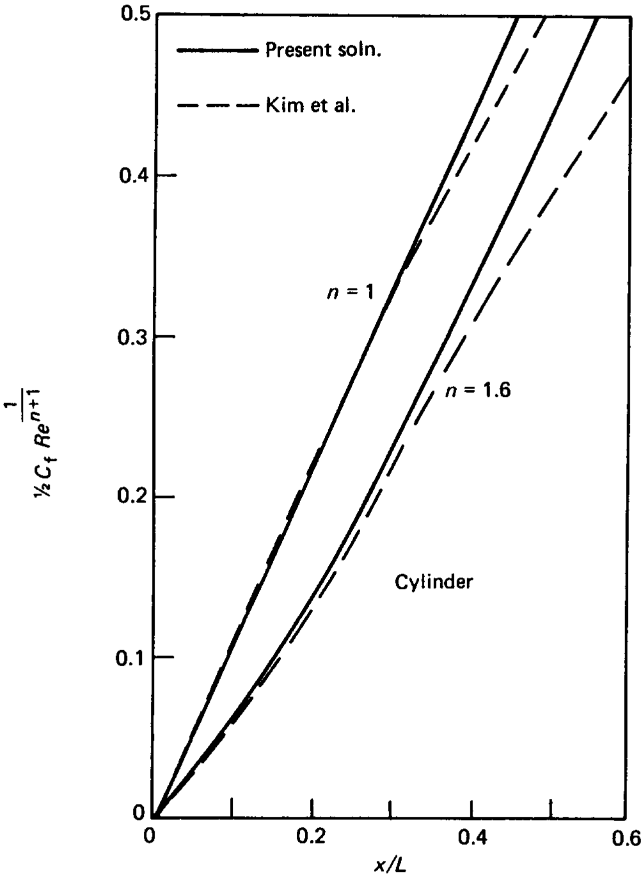

Fig 1 Local skin friction coefficient

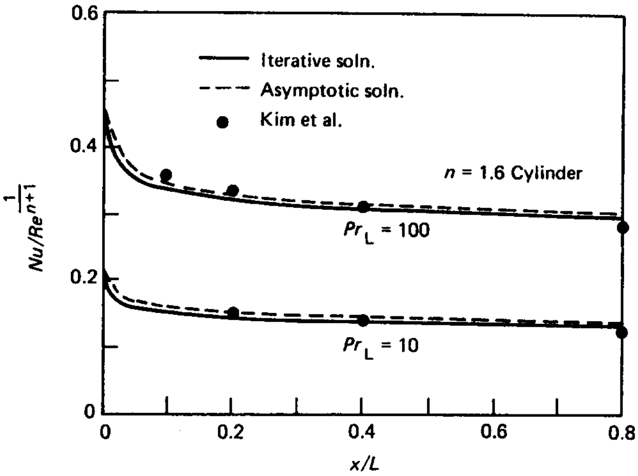

Fig 2 Local heat transfer coefficient

The foregoing equation suggests $m_t = (2+n-3m)/3(1+n)$ under the constant wall heat flux condition.

Results and discussions

For the case of an isothermal flat plate ($m = m_t = 0$), Eq (11) along with Eq (5) gives

$$Nux/Rex^{1/(1+n)}Prx^{1/3} = \left[\frac{8}{45}\frac{1+2n}{1+n}\left(\frac{37}{630(1+n)2^{n-1}}\right)^{1/(1+n)}\right]^{1/3} \quad (12)$$

The foregoing equation, for example, predicts $Nux/Rex^{1/(1+n)}Prx^{1/3} = 0.3253$ at $n = 0.5$, and 0.3808 at $n = 1.5$, which are very close to the values (namely, 0.3254 and 0.3566) obtained by Acrivos *et al*[2] using a combined analytical-numerical method.

It is also of great interest to investigate the local heat transfer around a 2-D stagnation point, where the free stream velocity increases in proportion to the distance x, measured along the wall surface from the stagnation point. Kim *et al*[5] analysed a stagnation flow over a horizontal isothermal circular cylinder employing the Merk-type of series expansion technique. The external velocity field was assumed to follow the empirical formula given by Shah *et al*[6]:

$$u_e/u_\infty = 0.92(x/L) - 0.131(x/L)^3 \quad (13)$$

where u_∞ is the uniform approaching flow velocity while L is the radius of the cylinder. Upon approximating u_e/u_∞ by retaining only the first term on the right hand side of Eq (13), the local skin friction and heat transfer rate have been evaluated according to Eqs (6) and (11). The results are presented in Figs 1 and 2, where Re, C_f and Nu appearing in ordinate variables are defined as $Re = \rho u_\infty^{2-n}L^n/K$, $C_f = C_{fx}(u_e/u_\infty)^2$ and $Nu = Nux(L/x)$, respectively. Reasonably good agreement may be observed between the present asymptotic results and those by Kim *et al*[5]. Heat transfer results obtained by the integration step-wise iterative procedure[4] are also indicated in Fig 2. A comparison of the iterative solution and the asymptotic solution reveals the validity of the asymptotic expression derived in this study.

Conclusions

A general asymptotic expression based on the assumption of high Prandtl number has been derived through a simple integral analysis. The resulting expression has been found to be quite useful for a speedy and accurate estimation of the forced convective heat transfer from external surfaces to inelastic power-law fluids.

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